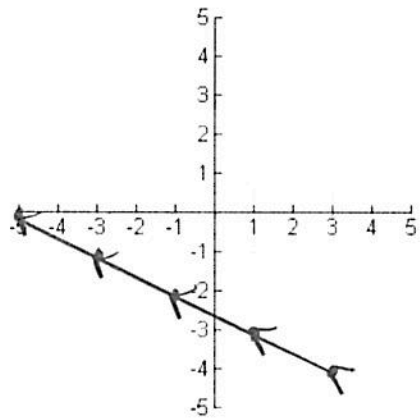
Notes: Parametric Equations – Day 1

As you walk around school each day, you probably cross paths with many of your friends. But does this necessarily mean that you meet them each time? What other information do you need to determine if the friends actually met?

Without knowing *when* the friends were at the intersections, you cannot determine if they met. There are many situations in which time is a major component. Parametric equations are equations that add a third parameter, usually time, to situation.

EX1: Graph the parametric equation $\begin{matrix} x_t = 3 - 2t \\ y_t = -4 + t \end{matrix}$ for the time interval $0 \le t \le 4$.

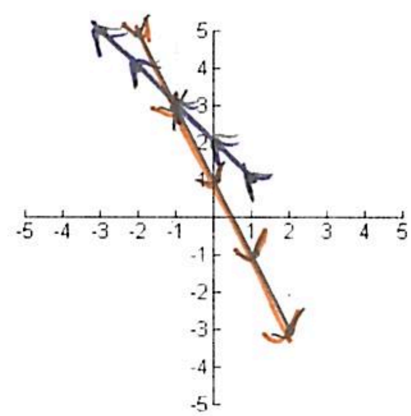
t	x =	y =
0	3	-4
1	1	-3
2	-1	-2
3	-3	-1
4	-5	0



EX2: Graph the parametric equations 1 $\begin{cases} x_t = -2 + t \\ y_t = 5 - 2t \end{cases}$ and 2 $\begin{cases} x_t = 1 - t \\ y_t = 1 + t \end{cases}$ on [0, 4].

different times

t	x_1 =	y_1 =	x_2 =	y_2 =
0	-2	5	1	1
1	-1	3	0	2
2	0	1	-1	3
3	1	-1	-2	4
4	2	-3	-3	5

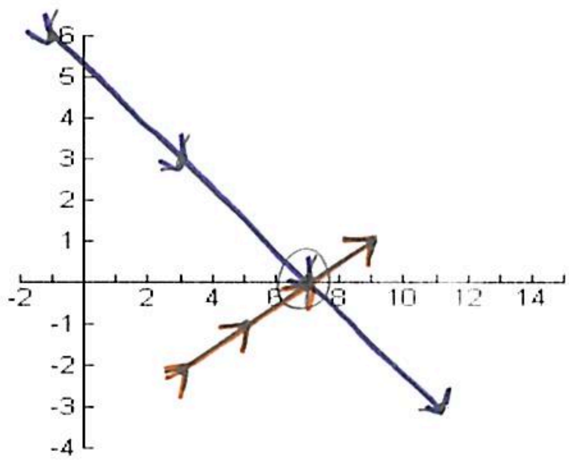


a) Do their lines of travel intersect? Yes

b) Do the two objects collide? NO If so, when? / at /

EX3: Graph the parametric equations 1 $\begin{cases} x_t = 3 + 2t \\ y_t = -2 + t \end{cases}$ and 2 $\begin{cases} x = -1 + 4t \\ y = 6 - 3t \end{cases}$ on [0, 3].

t	x_1 =	y_1 =	x_2 =	y_2 =
0	3	-2	-1	6
1	5	-1	3	3
✓ 2	7	0	7	0
3	9	1	11	-3



a) Do their lines of travel intersect? Yes

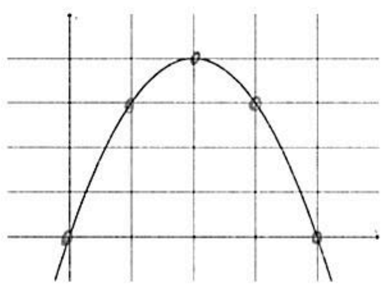
b) Do the two objects collide? Yes If so, when? t = 2 at (7, 0)

Parametric equations are not always lines. The path of a projectile can be represented by the function : $y = -(x-2)^2 + 4$. Its motion has both a horizontal and a vertical component. Horizontally, the projectile moves at a constant rate if air resistance is ignored. Vertically, it accelerates downward due to the action of gravity. Time is an important factor to consider in both cases.

We can use parametric equations to separately describe the x- and y- coordinates of a point. In a parametric equation, both coordinates will be written as a function of *t*, called the parameter.

Let's look at the table for the given function:

x	y
0	0
1	3
2	4
3	3
4	0



If we assume that the projectile was in the air for 4 seconds, we can write the following parametric equations:

$X_t = t$
$Y_t = -(t-2)^2 + 4$

t	X_t	Y_t
0	0	0
1	1	3
2	2	4
3	3	3
4	4	0

Making the x=T substitution is often the easiest way to convert between parametric and rectangular equations, but it is arbitrary, and not usually tied to actual time. We will often need negative t values for these types of problems.

Converting between Rectangular and Parametric:

It is easy to convert between rectangular and parametric equations. All we need to do is solve either equation for T and then substitute.

*Find the rectangular equations by eliminating the parameters:

a) $x_t = 2t+3$, $y_t = 3t-1$

$X = 2t + 3$

$X - 3 = 2t$

$t = \frac{x-3}{2}$

$y = 3\left(\frac{x-3}{2}\right) - 1 \rightarrow y = \frac{3x}{2} - \frac{9}{2} - \frac{2}{2} \rightarrow \boxed{y = \frac{3}{2}x - \frac{11}{2}}$

b) $x_t = t+2$, $y_t = t^2$

$x = t + 2$

$x - 2 = t$

$y = (x-2)^2 \rightarrow \boxed{y = x^2 - 4x + 4}$

While the easiest way to convert from rectangular to parametric is to let x = T, we can let T be just about anything.

*Find a set of parametric equations for each rectangular equation using t = 2 – x. $\rightarrow X = 2 - t$

a) $y = 3x - 2$

$y = 3(2-t) - 2$

$y = 6 - 3t - 2$

$= 4 - 3t$

$\boxed{\begin{aligned} x_T &= 2 - T \\ y_T &= 4 - 3T \end{aligned}}$

b) $y = x^2 + 1$

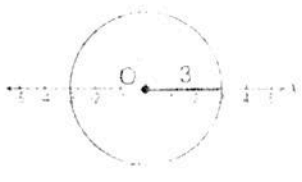
$y = (2-t)^2 + 1$

$= 4 - 4t + t^2 + 1$

$= t^2 - 4t + 5$

$\boxed{\begin{aligned} x_T &= 2 - T \\ y_T &= T^2 - 4T + 5 \end{aligned}}$

Circles and Ellipses as a Parametric Equation:



The equation of this circle can be written in 2 ways:

1) Using Conics

$x^2+y^2=9$, $y = \pm\sqrt{9-x^2}$

Let $x_t = t$

$y_t = \pm\sqrt{9-t^2}$

For our t min and t max, when x = t, we just want the t-values to fit the graph. -5<t<5

$x = r\cos t \pm h$ — part of center (r: x radius)

$y = r\sin t \pm k$ (r: y radius)

2) Using Trig

$x_t = 3\cos t$

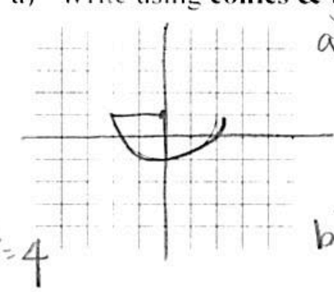
$y_t = 3\sin t$

$0 \le t \le 2\pi$

For our t min and t max when using trig, t is an angle. We need 0°<t<360° to go all the way around the circle.

Examples:

a) Write using **conics & trig**



center $(0,1)$

$r = 2$

$x^2 + (y-1)^2 = 4$

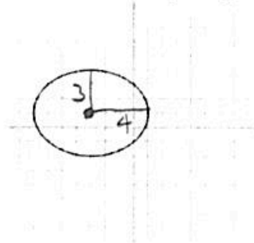
$\sqrt{(y-1)^2} = \pm\sqrt{4-x^2}$

$y = \pm\sqrt{4-x^2} + 1$ — use lower portion only

a) $\boxed{\begin{aligned} x_T &= T \\ y_T &= -\sqrt{4-T^2} + 1 \end{aligned}}$

b) $\boxed{\begin{aligned} x_T &= 2\cos t \\ y_T &= 2\sin t + 1 \end{aligned}}$

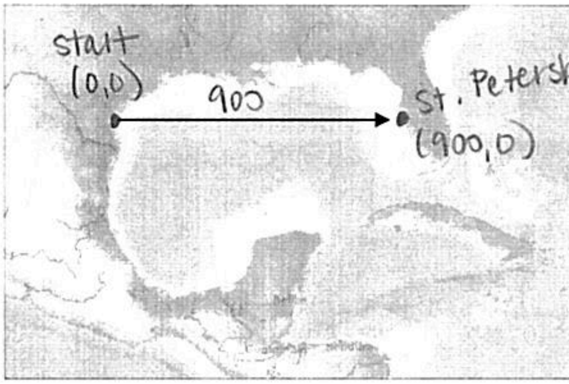
b) Write using **trig**



$\boxed{\begin{aligned} x_T &= 4\cos t - 3 \\ y_T &= 3\sin t + 1 \end{aligned}}$

Two tankers leave Corpus Christi at the same time, traveling toward St. Petersburg, which is 900 miles east. Tanker A travels at 18 mph and Tanker B travels at 22 mph. Establish a coordinate system and use your calculator to model the motion. Then answer the questions that follow.

$D = rt$



Tanker A:

$X_t = 18t$

$Y_t = 1$

Tanker B:

$X_t = 22t$

$Y_t = 2$

y-values just show different paths on graph for the tankers

The origin will be at Corpus Christi.
For the window we will use $0 \leqslant t \leqslant 50$ (900/18), $0 \leqslant x \leqslant 900$, and $0 \leqslant y \leqslant 3$.

***Make sure that under mode you are in <u>simultaneous</u> so that the tankers will start at the same time.

a) How long does it take the faster tanker to reach St. Petersburg?

$\frac{900}{22} = \boxed{40.909 \text{ hours}} - t$

b) Where is the slower tanker when the faster tanker reaches its destination?

$X = 18(40.909) = \boxed{736.362 \text{ miles east of C.C}}$

c) When during the trip is the faster tanker exactly 82 miles in front of the slower tanker?

$\longrightarrow 22T$

$\longrightarrow 18T$ 82

$22T = 18T + 82$

$\frac{4T}{4} = \frac{82}{4}$ $\boxed{T = 20.5 \text{ hours}}$

d) During what part of the trip are the tankers less than 60 miles apart?

$22T = 18T + 60$

$4t = 60$

$T = 15$

$\boxed{0 \leq T \leq 15 \text{ hours}}$