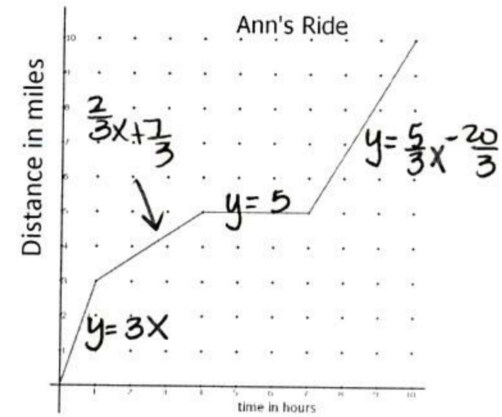
Key

Continuity and Limits Packet 2019

Lesson 1: Piecewise Functions

A piecewise function has different rules for different parts of its domain.

1. Ann went on a bicycle trip. The graph shows the relationship between time and distance traveled by Ann.



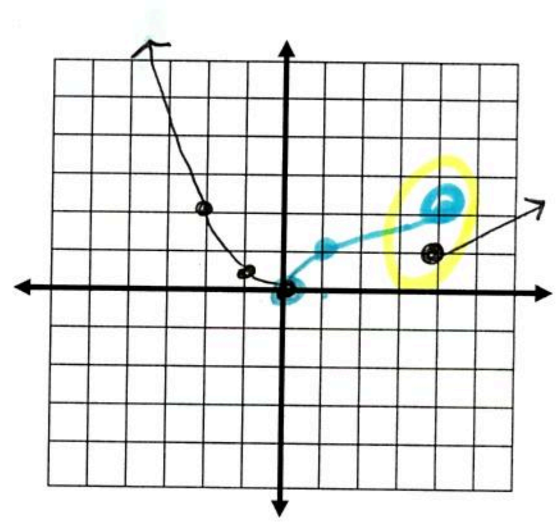
a. Write a function $d(t)$ for her distance in miles traveled in terms of the time (t) in hours.

$$d(t)=\begin{cases} 3t, & 0 \le t \le 1 \\ 2/3\,t + 7/3, & 1 < t \le 4 \\ 5, & 4 < t \le 7 \\ 5/3\,t - 20/3, & 7 < t \le 10 \end{cases}$$

b. Find $d(3)$. $\frac{2}{3}(3)+\frac{7}{3} = \boxed{\frac{13}{3}}$

2. Graph the following piece-wise function and state its domain and range.

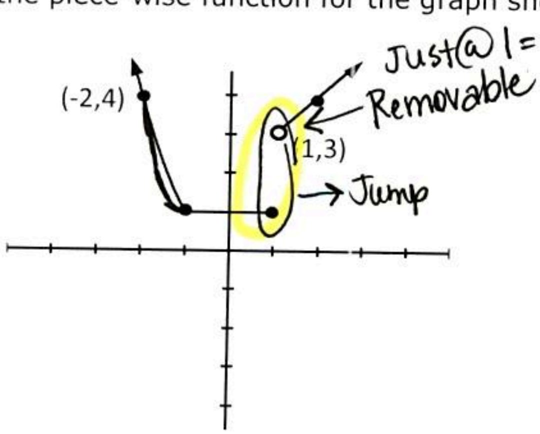
$$f(x)=\begin{cases} \frac{1}{2}x^2 & x \le 0 \\ \sqrt{x} & 0 < x < 4 \\ \frac{x}{4} & x \ge 4 \end{cases}.$$



Evaluate: $f(0)=0$ $\quad f(4)=1$ $\quad f(1)=1$

D: $(-\infty,\infty)$ $\mathbb{R}$ R: $[0,\infty)$

3. Write the piece-wise function for the graph shown.



$$f(x)=\begin{cases} x^2, & x \le -1 \\ 1, & -1 < x \le 1 \\ x+2, & x > 1 \end{cases}$$

Continuity of Functions

Continuous - The function is a single unbroken curve. You can trace the function from left to right without lifting your pencil.

Discontinuous - This is the result if there is one or more discontinuity.

Rat'l

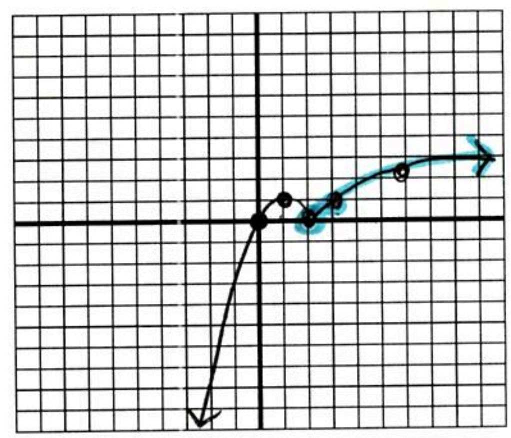
- Removable Discontinuity (hole) - the function would be continuous in the absence of the hole. if you "plug" the hole, the function becomes continuous.
- Jump Discontinuity - At a certain domain value, the function's value "jumps" and then continues. The jump creates a break in the curve where you must lift your pencil to trace it.
- Infinite Discontinuity - A vertical asymptote interrupts the continuity of the function.

Sketch the graph of each function, then identify the types of discontinuities (if any).

4. $f(x)=\begin{cases}1-(x-1)^2, & x \le 2 \\ \sqrt{x-2}, & x>2\end{cases}$ →

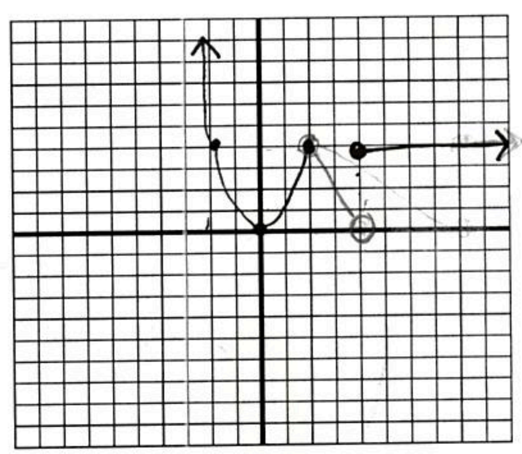
0	0
1	1
2	0

2	0
3	1



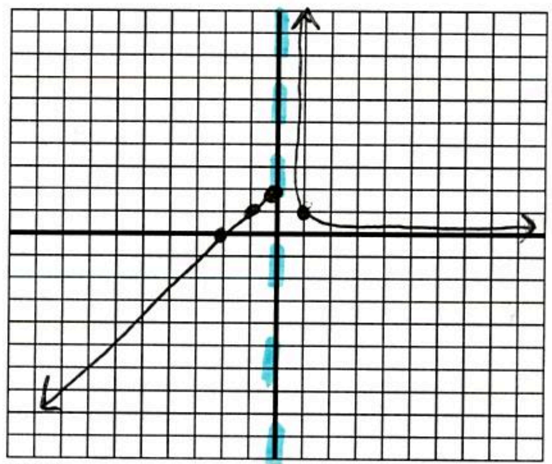
Discontinuities: none

5. $f(x)=\begin{cases}x^2, & x \le 2 \\ 8-2x, & 2<x<4 \\ 4, & x \ge 4\end{cases}$



Discontinuities: jump at x=4

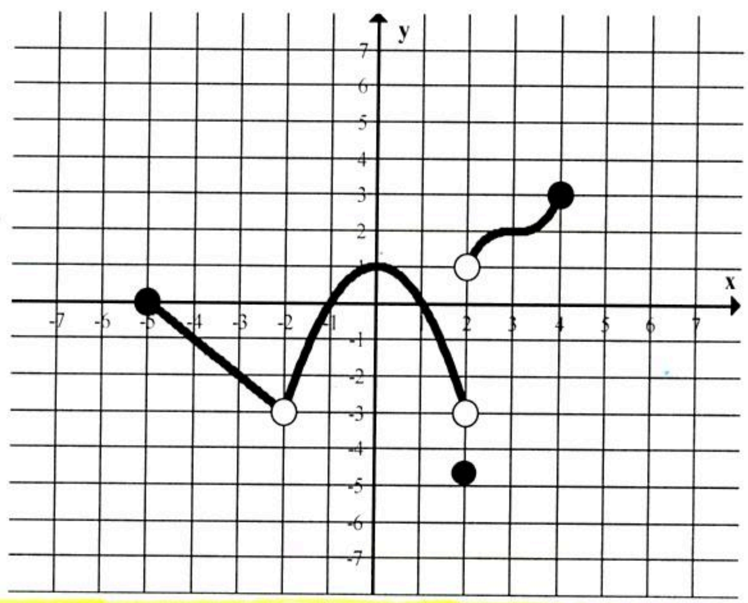
6. $f(x) = \begin{cases} x+2, & x \le 0 \\ \frac{1}{x}, & x > 0 \end{cases}$

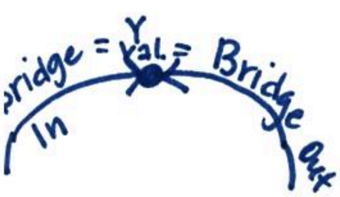


Discontinuities: unfinite @ X=0 from Rt. (vertical asy.)

7. Write a piecewise defined function for the following graph.

$f(x) = \begin{cases} -x-5, & -5 \le x < -2 \\ -x^2+1, & -2 < x < 2 \\ -5, & x = 2 \\ (x-3)^3+2, & 2 < x \le 4 \end{cases}$





In order for a function to be continuous at a point,

a) f(a) must exist

b) $\lim_{x \to a} f(x)$ **must exist** * talk about next class

c) f(a)= $\lim f(x)$

Where is this function discontinuous within the domain [-5,4]? @ X=-2 and X=2

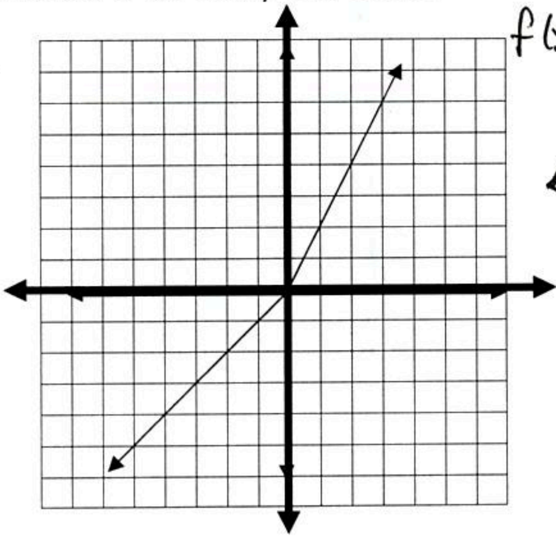
Where is this function continuous within the domain [-5,4]? [-5,-2) ∪ (-2,2) ∪ (2,4]

Now, go back to **4,5,6,7** and state their intervals of continuity.

Piecewise Functions Homework

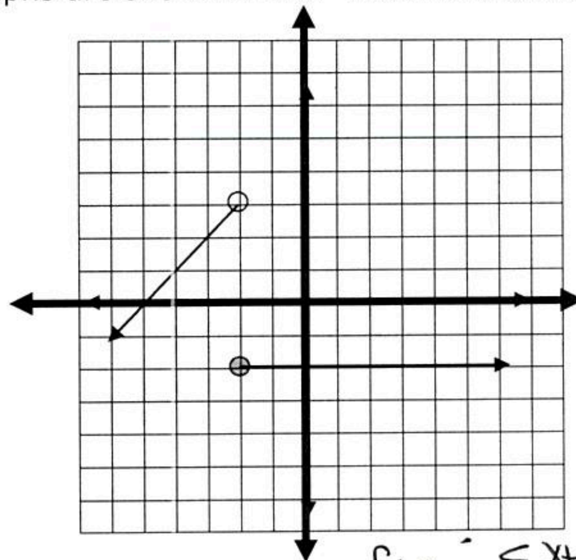
Write equations for the piecewise functions whose graphs are shown below. Assume that the units are 1 for every tick mark.

1\.



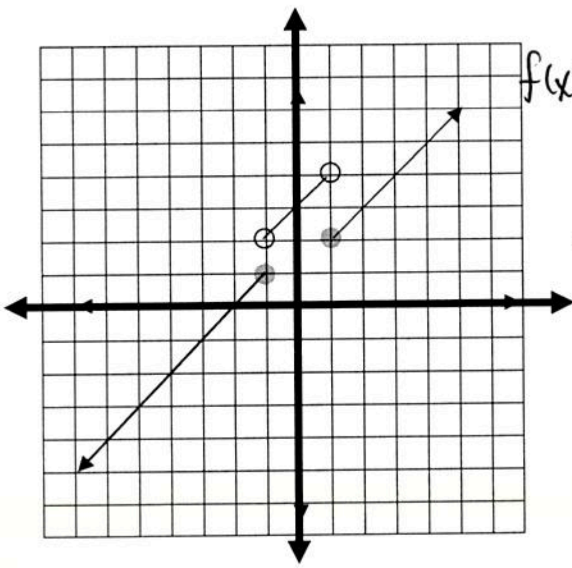
$f(x) = \begin{cases} x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

2\.



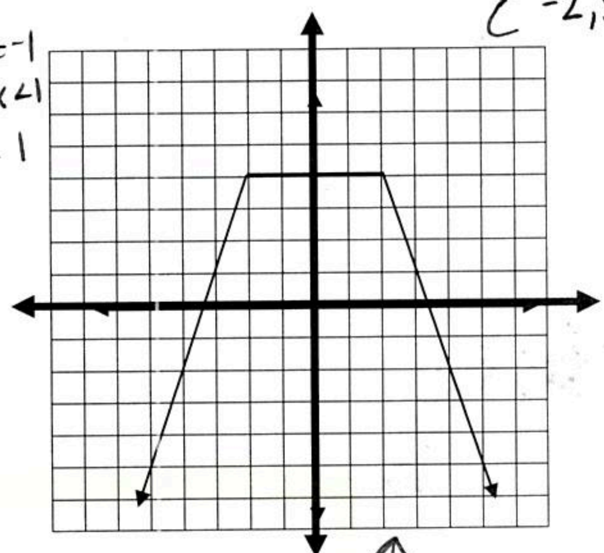
$f(x) = \begin{cases} x+5, & x \le -2 \\ -2, & x \ge -2 \end{cases}$

3\.



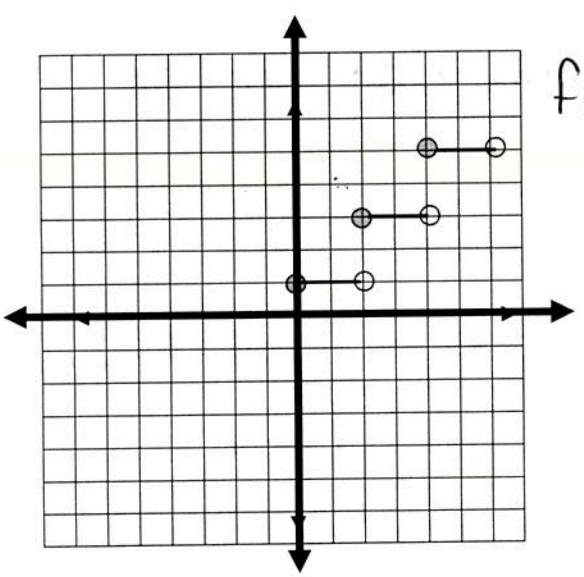
$f(x) = \begin{cases} x+2, & x \le -1 \\ x+3, & -1 < x < 1 \\ x+1, & x \ge 1 \end{cases}$

4\.



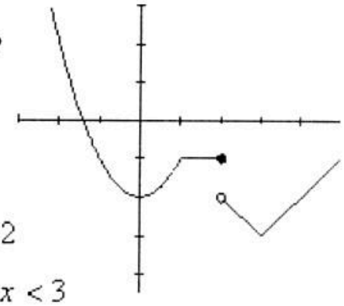
$f(x) = \begin{cases} 3x+10, & x \le - \\ 4, & -2 < x < \\ -3x+10, & x \ge \end{cases}$

5\.



$f(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3, & 2 \le x < 4 \\ 5, & 4 \le x < 6 \end{cases}$

6. Which of the following piecewise functions is represented by this graph?



a. $f(x)=\begin{cases} x^2-2 & x\le 1 \\ -1 & 1<x<2 \\ |x-3|-3 & x\ge 2 \end{cases}$

c. $f(x)=\begin{cases} x^2-2 & x\le -2 \\ -1 & -2<x<3 \\ |x-3|-3 & x\ge 3 \end{cases}$

b. $f(x)=\begin{cases} x^2-2 & x\le -2 \\ -1 & -2<x\le 3 \\ |x-3|-3 & x>3 \end{cases}$

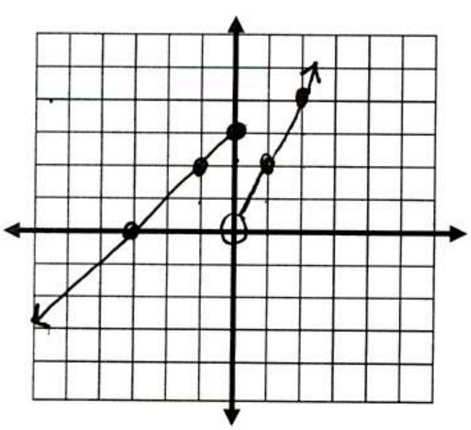
d. $f(x)=\begin{cases} x^2-2 & x\le 1 \\ -1 & 1<x\le 2 \\ |x-3|-3 & x>2 \end{cases}$

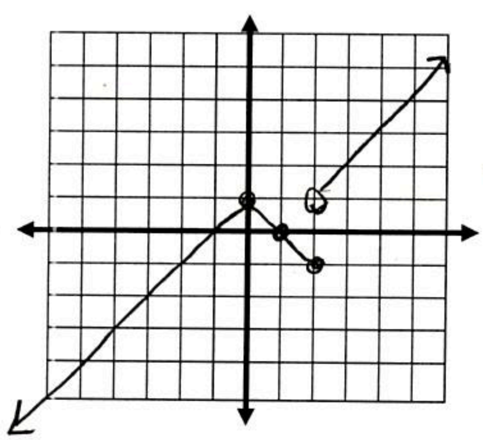
Graph the function.

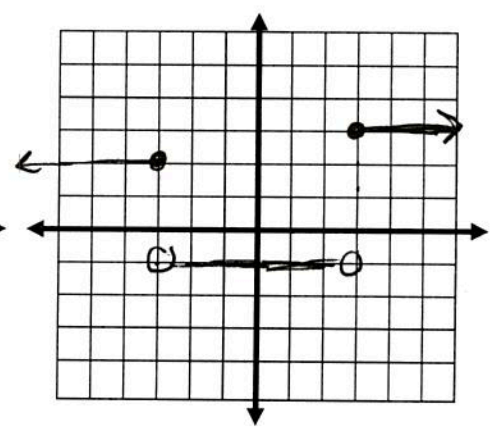
7. $f(x)=\begin{cases} x+3, \text{ if } x\le 0 \\ 2x, \text{ if } x>0 \end{cases}$

8. $f(x)=\begin{cases} x+1, \text{ if } x<0 \\ -x+1, \text{ if } 0\le x\le 2 \\ x-1, \text{ if } x>2 \end{cases}$

9. $f(x)=\begin{cases} 2, \text{ if } x\le -3 \\ -1, \text{ if } -3<x<3 \\ 3, \text{ if } x\ge 3 \end{cases}$







STOP

LIMITS Day 1

Estimate a Limit Numerically (Explore)

What is a limit?

Informally: As the x value gets closer and closer to some number c **from both sides**, the **y value** gets closer and closer to some number L. The "L" is the limit. (*-y value)

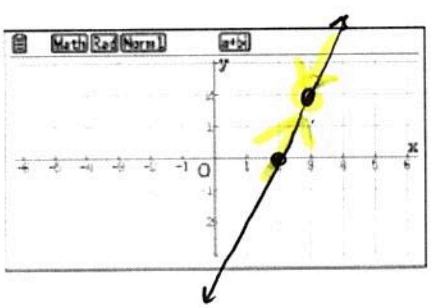
Notation: $\lim_{x \to c} f(x) = L$

1. Estimate $\lim_{x\to 3}(2x-4)$ by

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	1.8	1.98	1.998	2	2.002	2.02	2.2

$\boxed{\lim_{x\to 3}(2x-4)=2}$

Now sketch graph below.



Would $f(x)$ be continuous at $x = 3$?

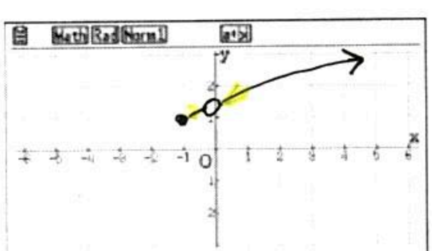
$\lim_{x\to 3^-}$ ✓ $\lim_{x\to 3^+}$ ✓ $f(3) =$ ✓

Yes!

2. Estimate $\lim_{x\to 0}\left(\frac{x}{\sqrt{x+1}-1}\right)$ by

x	-.01	-.001	-.0001	0	.0001	.001	.01
$f(x)$	1.995	1.999	1.999	Error	2.000	2.0005	2.005

Now sketch graph below.



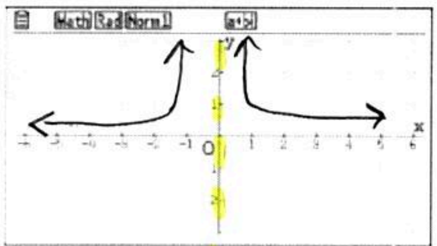
$\boxed{\lim_{x\to 0}\left(\frac{x}{\sqrt{x+1}-1}\right)=0}$

3. Estimate $\lim_{x\to 0}(\frac{1}{x^2})$

✩ notice symmetry

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	100	10000	~~Error~~ 1E6	~~1E6~~ Err	1000 1E6	10000	100

Now sketch graph below,



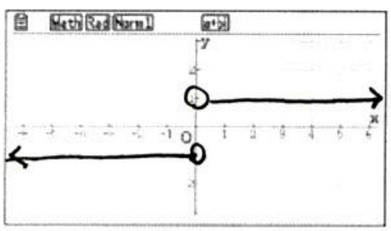
← VA

Since $\lim_{x\to 0^-} = \infty$, $\lim_{x\to 0^+} = \infty$,

the lim is $= \infty$ or DNE is an appropriate answer

Continuous at $x=0$? NO!

4. Left and Right Hand Behavior: Investigate the graph of $f(x)=\frac{|x|}{x}$. For what value is $f(x)$ undefined? $x =$ 0



This tells you where the limit ***MIGHT*** fail to exist.

Investigate the limit using a table as x values approach this value.

x	-2	-1	-0.5	0	0.5	1	2
$f(x)$	-1	-1	Error	1	1	1	1

What does the graph appear to be approaching from the left? $x \to 0^- =$ -1.

What does the graph appear to be approaching from the right? $x \to 0^+ =$ +1.

What does this imply about $\lim_{x\to 0}\frac{|x|}{x}$? DNE Why? Because the limits from ea. side are not the same ∴ lim DNE.

Limit formal definition:

If the value of a function becomes close to a <u>unique</u> number L as x approaches a number c from <u>both sides</u>, the limit of the function as x approaches c is L.

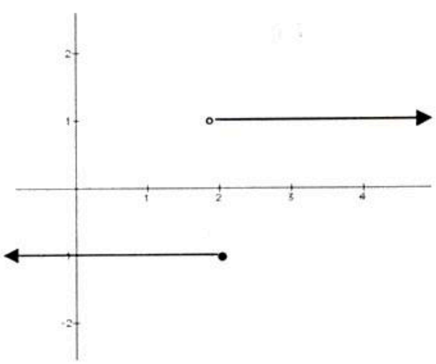
$$\lim f(x) = L$$

$\lim_{x\to c^-} f(x) =$ the limit from the left

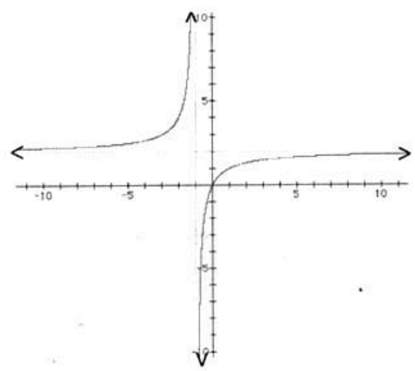
$\lim_{x\to c^+} f(x) =$ the limit from the right

When $\lim_{x\to c}$ **f(x)** DOES NOT EXIST – The limit of a function as x approaches c does not exist if :

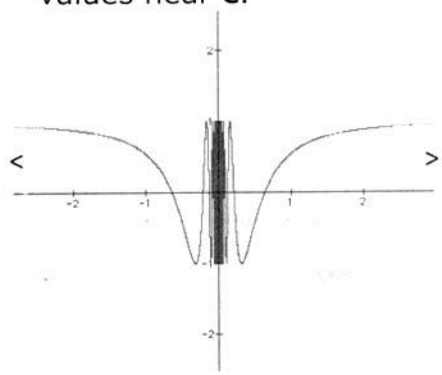
*The function approaches different values from the left and right sides of **c**.



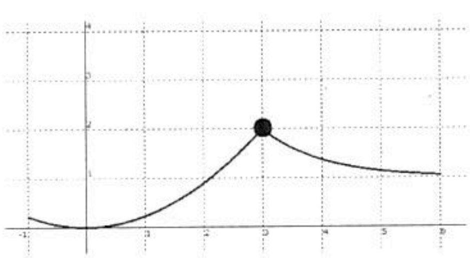
*The function has a a vertical asymptote at **c**.



*The function oscillates between 2 different values near **c**.



<u>Finding a limit graphically:</u>

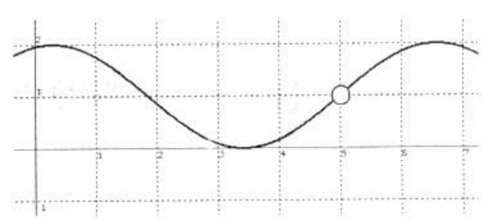


$\lim_{x\to 3^-} f(x) =$ 2

$\lim_{x\to 3^+} f(x) =$ 2

f(3) = 2

$\lim_{x\to 3} f(x) =$ 2

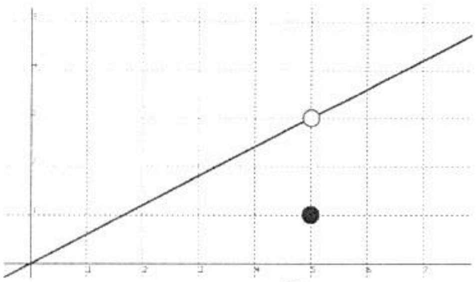


$\lim_{x\to 5^-} f(x) =$ 1

$\lim_{x\to 5^+} f(x) =$ 1

f(5) = Dne

$\lim_{x\to 5} f(x) =$ 1

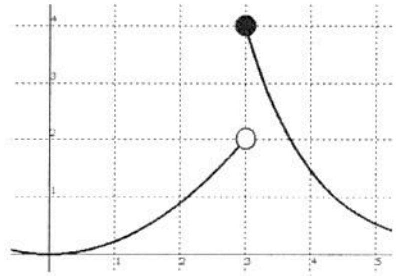


$\lim_{x\to5^-} f(x) =$ 3

$\lim_{x\to5^+} f(x) =$ 3

f(5) = $\cancel{8}$ 1

$\lim_{x\to5} f(x) =$ 3



$\lim_{x\to3^-} f(x) =$ 2

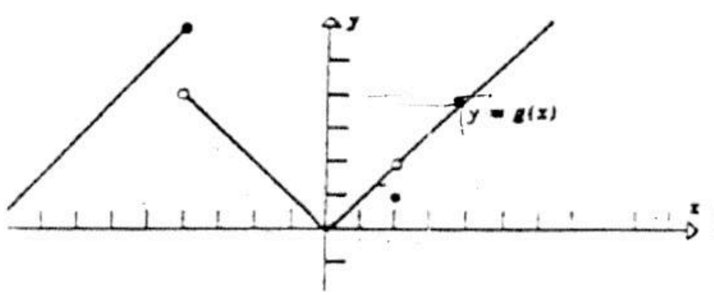
$\lim_{x\to3^+} f(x) =$ 4

↑ diff from left & right

f(3) = 4

$\lim_{x\to3} f(x) =$ dne

Find the limits using the piecewise function.



$\lim_{x\to-4^-} g(x) =$ 4

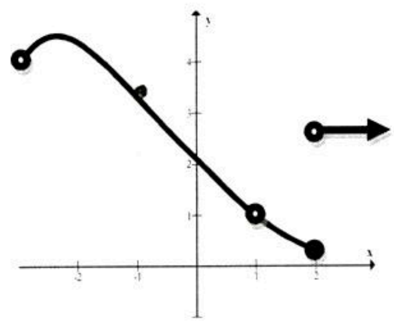
$\lim_{x\to-4^+} g(x) =$ 4

$\lim_{x\to2} g(x) =$ 2

$g(2) =$ 1

Limits Introduction
Homework

1. The graph of f(x) is shown in the figure below. Which of the following statements about f(x) is true?



A) $\lim_{x\to 1} f(x) = DNE$ F

B) $\lim_{x\to 2} f(x) = 0.3$ F

C) $\lim_{x\to 2.01} f(x) < \lim_{x\to 0} f(x)$ F

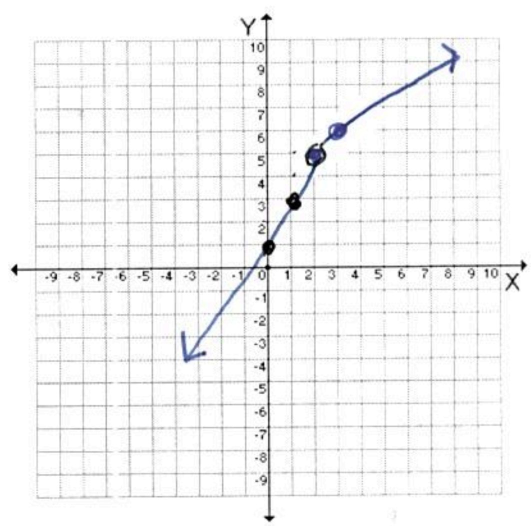
D) $\lim_{x\to -1} f(x) \approx 3.3$ T

E) $\lim_{x\to -3} f(x) = 4$ F

2. Graph the function and find the limit (if it exists) as x approaches 2.

$$f(x) = \begin{cases} 2x+1, & x<2 \\ x+3, & x\geq 2 \end{cases}$$

$\lim_{x\to 2} f(x) =$ 5



#3 Estimate the limit by using a table.

(Calculator permitted) Fill in the table for the following function, then use the numerical evidence (to 3 decimal places) to evaluate the indicated limit. (Be sure you're in radian mode)

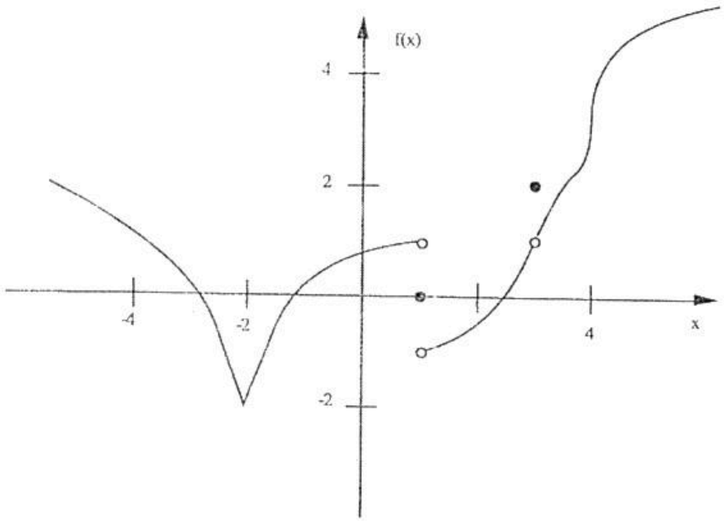
$$f(x) = \frac{\sin(3x)}{x}$$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	2.9552	2.999	3	Error	3	2.999	2.9552

Based on the numeric evidence above, $\lim_{x\to 0} f(x) =$ 3

4.

Given the graph of the following function.



Find the following:

A) $f(1)$ $= 0$

B) $\lim_{x\to 1^-} f(x)$ $= 1$

C) $\lim_{x\to 1^+} f(x)$ $= -1$

D) $\lim_{x\to 1} f(x)$

Dne

E) $f(3)$ $= 2$

F) $\lim_{x\to 3^-} f(x)$ $= 1$

G) $\lim_{x\to 3^+} f(x)$ $= 1$

H) $\lim_{x\to 3} f(x)$ $= 1$

I) $\lim_{x\to -2} f(x)$ $= -2$

J) $\lim_{x\to 0} f(x)$ $\approx .9$

K) $\lim_{x\to -3} f(x)$ $\approx .1$

Lesson 3: Algebraic Techniques for Finding Limits

Limit Rules

Limit of a constant	$\lim_{x\to\#} constant$ = constant	
Limit of a Sum/Difference	$\lim_{x\to\#}[f(x) \pm g(x)]$	$= \lim_{x\to\#} f(x) \pm \lim_{x\to\#} g(x)$
Limit of a Product	$\lim_{x\to\#}[f(x)\cdot g(x)]$	$= \lim_{x\to\#}(f(x)) \cdot \lim_{x\to\#}(g(x))$
Limit of a Quotient	$\lim_{x\to\#} \frac{f(x)}{g(x)}$	$= \frac{\lim_{x\to\#} f(x)}{\lim_{x\to\#} g(x)}$ provided $\lim_{x\to\#} g(x) \neq 0$ and $g(x) \neq 0$

Examples:

1. $\lim_{x\to c} 7 =$ 7
2. $\lim_{x\to c} x =$ c
3. $\lim_{x\to 3}(x+4) =$ 7
4. $\lim_{x\to 2}(x-5) =$ -3
5. $\lim_{x\to 3}(-5x) =$ -15
6. $\lim_{x\to 6} \frac{x}{2} =$ 3
7. $\lim_{x\to 3}(5x^2) =$ 45
8. $\lim_{x\to 2}(3x^4 - 5x^3 + 2x^2 - 6) =$ 10

Techniques for Evaluating Limits

Remember, if the function is continuous at the value indicated, use $\boxed{\text{direct substitution.}}$

If the function is NOT continuous at the value indicated:

Try 1) Factoring and reducing, 2) Long or synthetic division, 3) multiplying by the conjugate. 4) simplifying a complex expression, 5) using trig identities, or if all else fails, 6) sketch a graph of the function.

I. Dividing Out:

Factor and cancel where applicable. If you cannot factor it, then use synthetic or long division. Then use direct substitution on the simplified form (or quotient).

Ex 1: $\lim_{x\to 2} \frac{(x^3-2x^2)(x+2)}{x^4-2x^3+x-2}$ $x^2(x-2) - (x-2)$

$$\boxed{\lim_{x\to 2}\left[\frac{(x^2-1)\cancel{(x-2)}}{(x^3+1)\cancel{(x-2)}}\right] = \frac{3}{9} = \boxed{\frac{1}{3}}}$$

Ex 2: $\lim_{x\to 4} \frac{2x^3-3x^2-23x+12}{x-4}$ If I plug in immediately, I get $\frac{}{0}$ - won't work.

$$\begin{array}{r|rrrr} 4 & 2 & -3 & -23 & 12 \\ & \downarrow & 8 & 20 & -12 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$\rightarrow \boxed{\lim_{x\to 4}(2x^2+5x-3) = \boxed{49}}$

II. Rationalizing Technique: Multiply by the conjugate to rationalize either the numerator **or** the denominator.

Ex 3: $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{(\sqrt{x}+2)}{(\sqrt{x}+2)} = \lim_{x\to 4}\left(\frac{x-4}{(x-4)(\sqrt{x}+2)}\right) = \lim_{x\to 4}\left(\frac{1}{\sqrt{x}+2}\right)$

$= 0$

$= \lim_{x\to 4}\left(\frac{1}{\sqrt{x}+2}\right) = \boxed{\frac{1}{4}}$

III. Using Algebra: Sometimes you have to use algebra and manipulate things:

Ex 4: $\lim_{x\to 2} \frac{\frac{1}{x+2}-\frac{1}{4}}{x-2}$

$4(1) \quad (x+2)$

$= \lim_{x\to 2}\left(\frac{4-x-2}{4(x+2)}\right)\left(\frac{1}{x-2}\right)$

$= \lim_{x\to 2}\left(\frac{-1}{4(x+2)}\right) = \boxed{-\frac{1}{16}}$

IV. Trig When you see a trig function, and direct substitution does not work, try using identities:

Ex 5: $\lim_{x\to 0}(sin^2x + cos^2x)$ = $\boxed{1}$

$\lim_{x\to 0}(1) = \boxed{1}$

Ex 6: $\lim_{x\to 0}(\csc x \sin x)$= $\boxed{1}$

$\to \frac{1}{\cancel{\sin x}} \cdot \cancel{\sin x}$

Ex 7: Given $f(x)=\begin{cases} 2-x & x<1 \\ 2x-x^2 & x>1 \end{cases}$, find the following:) piecewise

$2(2)-2^2$

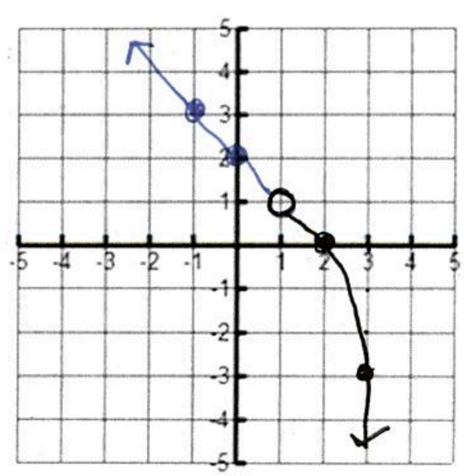
$2(3)-9$

$\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1} f(x) = 1$

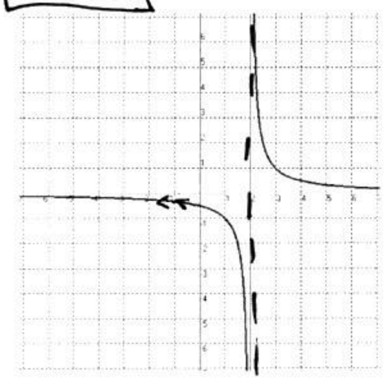
could also plug in



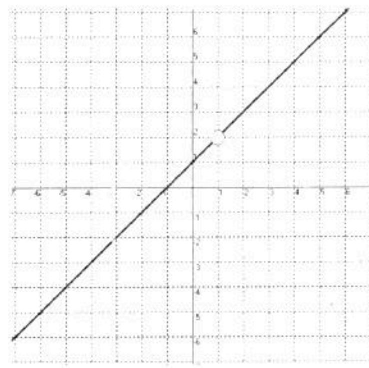
Piecewise and Limits Practice

Use the graphs to find the limits if they exist. If not, write DNE.

1. a) $\lim_{x\to 2}\left(\frac{1}{x-2}\right) = \boxed{\text{dne}}$ b) $\lim_{x\to -1}\left(\frac{1}{x-2}\right) = \boxed{-\frac{1}{3}}$



2. a) $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) = \boxed{2}$ b) $\lim_{x\to -5}\left(\frac{x^2-1}{x-1}\right) \boxed{-4}$
c) $\lim_{x\to 0}\left(\frac{x^2-1}{x-1}\right) = \boxed{1}$



$= \frac{1-x-2}{x+2} \cdot \frac{1}{x+1} \rightarrow \frac{-\cancel{(1+x)}}{x+2} \cdot \frac{1}{\cancel{x+1}} \rightarrow \frac{-1}{x+2}$

Find the limit, if it exists.

3. $\lim_{x\to 4}\left(\frac{1}{2}x+3\right)$

$= \boxed{5}$

4. $\lim_{x\to 3}\frac{|x-3|}{x-3}$

$\boxed{\text{Dne}}$

5. $\lim_{x\to -1}\left(\frac{\frac{1}{x+2}-1}{x+1}\right)\frac{x+2}{x+2}$

$\frac{\frac{1}{1}-\cancel{1}}{\cancel{-1}+1}$

$\lim_{x\to -1} \frac{-1}{x+2} = \boxed{-1}$

6. $\lim_{x\to -3}\left(x^3-6x^2+3x-1\right)$

$(-3)^3-6(-3)^2+3(-3)-1$

$= \boxed{-91}$

7. $\lim_{x\to 5}\left(\frac{x-5}{x^2+5x-50}\right)$

$\lim_{x\to 5} \frac{\cancel{x-5}}{\cancel{(x-5)}(x+10)} \cdot 1$

$\lim_{x\to 5}\left(\frac{1}{x+10}\right) = \boxed{\frac{1}{15}}$

8. $\lim_{x\to 0}\left(\frac{\sqrt{x+4}-2}{x}\right) \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$

$\rightarrow \frac{\cancel{x+4-4}}{\cancel{x}(\sqrt{x+4}+2)}$

$\rightarrow \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \boxed{\frac{1}{4}}$

9. ~~$\lim \frac{4x}{2x-3}$~~

10. $\lim_{x\to 2} f(x)$ where $f(x) = \begin{cases} 5-x, & x\le 2 \\ x^2-3, & x>2 \end{cases}$

$\lim_{x\to 2^-}(5-x) = 3$

$\lim_{x\to 2^+}(x^2-3) = 1$

$\boxed{\therefore \text{Lim Dne}}$

11. $\lim_{x \to -2} \frac{x^3 - 2x^2 - x + 14}{x+2}$

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -1 & 14 \\ & & -2 & 8 & -14 \\ \hline & 1 & -4 & 7 & \emptyset \end{array}$$

$\lim_{x \to -2} x^2 - 4x + 7 = \boxed{19}$

12. $\lim_{x \to \frac{3\pi}{4}} (\csc x)$

$\frac{1}{\sin} \quad \frac{\sqrt{2}}{2}$

$\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\cancel{2}\boxed{\sqrt{2}}}{\cancel{2}}$

13. $\lim_{x \to \frac{\pi}{2}} (\tan x)$

0, 1

$= \boxed{\text{Dne}}$

$\lim_{x \to \frac{\pi}{2}} (\tan x) = \text{dne}$

14. $\lim_{x \to 5} \frac{x^3 - 125}{x-5}$

$\frac{\cancel{(x-5)}(x^2+5x+25)}{\cancel{x-5}}$

$\lim_{x \to 5} x^2 + 5x + 25 = \boxed{75}$

15. $\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16} \cdot \frac{(\sqrt{x}+4)}{(\sqrt{x}+4)} \Rightarrow \frac{\cancel{x-16}}{\cancel{(x-16)}(\sqrt{x}+4)}$

$\lim_{x \to 16} \frac{1}{\sqrt{x}+4} = \boxed{\frac{1}{8}}$

Evaluate the function for the given value of x.

$f(x) = \begin{cases} 3, if\ x \le 0 \\ 2, if\ x > 0 \end{cases}$ $\qquad g(x) = \begin{cases} x + 5, if\ x \le 3 \\ 2x - 1, if\ x > 3 \end{cases}$ $\qquad h(x) = \begin{cases} \frac{1}{2}x - 4, if\ x \le -2 \\ 3 - 2x, if\ x > -2 \end{cases}$

23. f(0)

$\boxed{3}$

24. f($\frac{1}{2}$)

$\boxed{2}$

25. g(-1)

$\boxed{4}$

26. g(3)

$\boxed{8}$

27. h(-2)

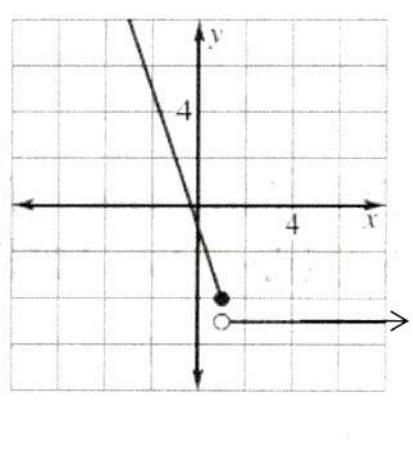
$\boxed{-5}$

28. h(6)

$\boxed{-9}$

29. Write a piecewise function for the following graph.

(notice increments on the axis)

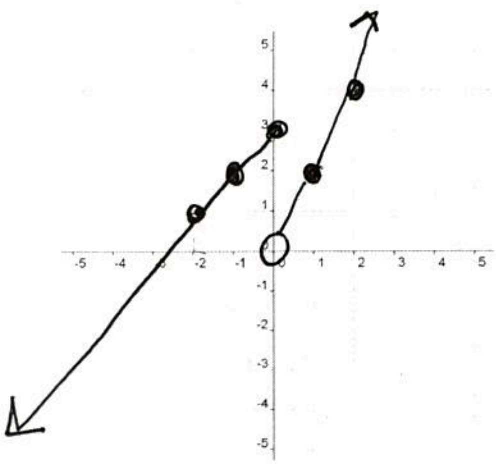


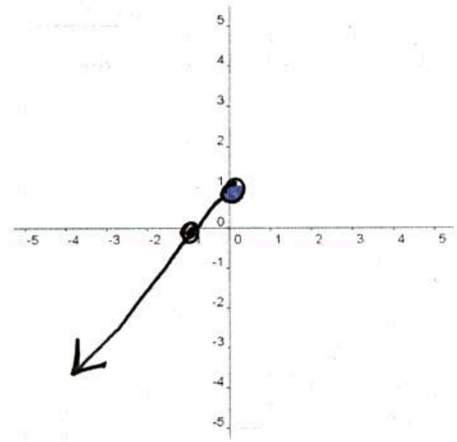
$\begin{cases} -3x - 1, & x \le 1 \\ -5, & x > 1 \end{cases}$

Graph the following.

30. $f(x) = \begin{cases} x+3, if\ x \le 0 \\ 2x, if\ x > 0 \end{cases}$

31. $f(x) = \begin{cases} x+1, if\ x < 0 \\ -x+1, if\ 0 \le x \le 2 \\ x-1, if\ x > 2 \end{cases}$





32. Is f(x) continuous at x=2? Why or why not (use correct notation) $f(x) = x^2 + 3$ → Prove.

$\lim_{x \to 2^-} x^2+3 = 7$, $\lim_{x \to 2^+} x^2+3=7$

$f(2) = 7$

Therefore, f(x) is continuous at x = 2.

33. Is f(x) continuous at x=5? Why or why not? $f(x) = \frac{x^3 - 125}{x-5}$ → Prove.

f(5) = undefined

Since f(5) isn't defined, the function is not continuous.

34. To find the limit of a function, must the function be continuous? Why or why not?

No. To be continuous, the $\lim_{x \to a} = f(a)$. To find the limit, only the