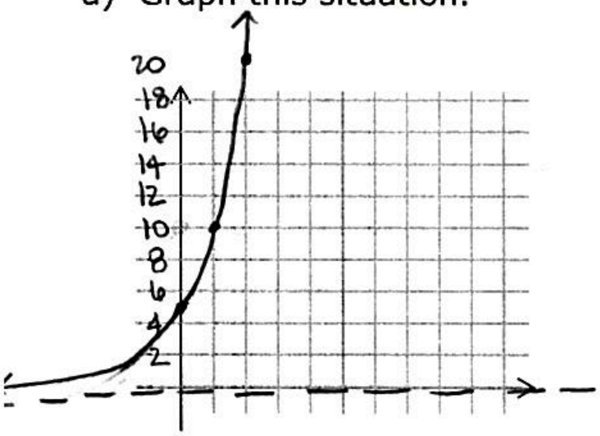
Notes on exponential growth models

Let's review: $\underbrace{y = \overset{\text{initial}}{a}(b)^{\overset{\text{time}}{x}}}_{}$ growth/decay factor (multiplier)

The amount of a substance doubles every day. If you started with 5 grams, how much will you have after 10 days?

a) Graph this situation.



b) Write a function that would represent this model. $y = 5(2)^x$

c) Solve.

$y = 5(2)^{10} = \boxed{5120 \text{ grams}}$

If we assume that the substance is growing continuously, then it should follow the growth model

$A(t) = Pe^{rt}$	P = initial amount	
A(t) = amount at time t	e^r= rate of growth/decay	t = time
r=continuous growth rate		

Since we are doubling,

t	A(t)
0	5
1	10
2	20
3	40

What if instead of doubling, we were cutting the amount in half. If it takes 5 days for the amount to cut in half, we say the half-life is 5 days. Let's start with 100 ounces of a substance with a half-life of 5 days. When will we have 20 ounces left?

t	A(t)
0	100
5	50
?	20

$$\frac{20}{100}=\frac{\cancel{100}\left(\frac{1}{2}\right)^{\frac{t}{5}}}{\cancel{100}}$$

$$0.2=0.5^{\frac{t}{5}}$$

$$0.5^{t/5}=0.2$$

$$(5)\ \log_{0.5}(0.2)=\frac{t}{\cancel{5}}\cancel{(5)}$$

$$\boxed{t=11.609\text{ days}}$$

The formula for half life is:

$$A(t)=P\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Where h=half-life.

Another example of continuous growth:

Money is invested in a bank that pays 5% interest compounded continuously. If Sam invested \$1000 in 1990, when would his money have doubled?

$r=0.05$ e P $t=?$

t	A(t)
1990 0	1000
?	2000

$$\frac{2000}{1000}=\frac{\cancel{1000}e^{0.05t}}{\cancel{1000}}$$

$$2=e^{0.05t}$$

$$e^{0.05t}=2 \rightarrow \frac{\ln 2}{0.05}=\frac{\cancel{0.05}t}{\cancel{0.05}}$$

$$\boxed{t=13.862\text{ years}}$$

Not all growth is continuous. When growth happens consistently at repeated intervals, we get a different formula.

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$$

r = annual interest rate

n = number of times it compounds in a year

t = number of years.

Example.

$r=0.05$ $n=12$

Money is invested in a bank that pays 5% interest compounded monthly. If Sam invested \$1000 in 1990, when would his money have doubled?

P $t=?$ $A(t)=2000$

	t	A(t)
1990	0	1000
	?	2000

$$\frac{2000}{1000}=\frac{\cancel{1000}\left(1+\frac{0.05}{12}\right)^{12t}}{\cancel{1000}}$$

$$2=\left(1+\frac{0.05}{12}\right)^{12t}$$

$$(1.004\ldots)^{12t}=2$$

don't round from calc!

$$\frac{\log_{1.004\ldots}(2)}{12}=\frac{12t}{12} \rightarrow \boxed{t=13.891 \text{ years}}$$

A few more examples:

continuous growth → Pe^{rt}

1) The bacteria in a certain culture increase according to the law of exponential growth.

If y = 3000 at the onset and y = 6000 when t = 5?

t	y
0	3000
5	6000

What is rate?

$$\frac{6000}{3000} = \frac{3000e^{r(5)}}{3000}$$

$$2 = e^{5r} \rightarrow e^{5r} = 2$$

$$\frac{\ln 2}{5} = \frac{5r}{5}$$

$$r = 0.138...$$

a. Find y when t = 1. $y = 3000e^{0.138...(1)} \approx \boxed{3446.095}$

b. Find t when y = 60,000.

$$\frac{60,000}{3000} = \frac{3000e^{0.138...t}}{3000} \rightarrow 20 = e^{0.138...t}$$

$$e^{0.138...t} = 20$$

$$\frac{\ln 20}{0.138...} = \frac{0.138...t}{0.138...}$$

$$\boxed{t = 21.609 \text{ yrs}}$$

2) The population of a town is decreasing according to the law of exponential decay. In 1995, the population was 50,000 and in 2005 it was 44,000. What is the expected population in 2015?

$$A(t) = P(1-r)^t$$

	t	P(t)
95	0	50,000
05	10	44,000
2015	20	?

what is r?

$$\frac{44,000}{50,000} = \frac{50,000(1-r)^{10}}{50,000}$$

$$0.88 = (1-r)^{10}$$

y_1 $(1-r)^{10} = 0.88$ y_2 → Graph and find intersection

$$r = 0.01270198$$

$$y = 50,000(1-0.012...)^{20}$$

$$\boxed{y = 38719.997 \text{ people}}$$

3) A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing according to the law of continuous exponential growth, then how much will the puppy weigh when it is 3 months old?

e

Months: t	w(t)
0	2
2	3.5
3	?

what is r?

$$\frac{3.5}{2} = \frac{2e^{r(2)}}{2}$$

$$1.75 = e^{2r}$$

$$e^{2r} = 1.75 \rightarrow \frac{\ln 1.75}{2} = \frac{2r}{2}$$

$$r \approx 0.279...$$

$$y = 2e^{0.279...(3)}$$

$$\boxed{y = 4.630 \text{ pounds}}$$